Titolo del corso: Mini Course on Universal Algebras in Stable Representation Theory: From Farahat-Higman to Ivanov-Kerov and Borodin–Olshanski

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Programma del corso: si veda allegato

Mini Course on Universal Algebras in Stable Representation Theory: From Farahat-Higman to Ivanov-Kerov and Borodin–Olshanski

This mini-course is devoted to a stability phenomenon in the representation theory of families of finite groups with the guiding example the sequence of symmetric groups S_n in mind. Although the centers

$$Z(\mathbb{Z}[S_n])$$

of these algebras change with n, their multiplication has a remarkable uniformity: after a suitable parametrization of conjugacy classes, the relevant structure constants become polynomial functions of n. Farahat and Higman used this polynomiality to construct a universal algebra encoding stable multiplication of conjugacy class sums.

The course follows this phenomenon through three related stages. First, we focus on the Farahat-Higman algebra as a universal object built from polynomial stability. Second, we shift our attention to the Ivanov-Kerov construction, where partial permutations concept is introduced and used as an intrinsic projective-limit realization of the same algebra. Third, we place this stable algebraic picture in the broader asymptotic representation theory of symmetric groups, following the first part of Borodin–Olshanski's book: finite symmetric groups, symmetric functions, coherent systems on the Young graph, and Thoma's theorem. Okounkov–Olshanski's theory of shifted symmetric functions will appear as the algebraic bridge between stable central elements and normalized characters.

The central theme is the passage

1. stable structure constants
2. universal algebra
3. projective-limit realization
4. asymptotic representation theory.

The course is aimed as an introduction to this circle of ideas, rather than a complete technical treatment of all proofs. Though, core constructions will be presented in detail, while the deeper asymptotic theory will be explained as the representation-theoretic environment in which the stable algebraic structures from the first two lectures naturally live.

Lecture 1. Farahat–Higman stability

We will review the original Farahat-Higman construction for the centers $Z(\mathbb{Z}[S_n])$. We begin with explicit products of conjugacy class sums, using the transposition class, in order to see how polynomial dependence on n appears concretely.

The second step will be to explain how Farahat and Higman used this polynomiality to define a universal algebra. This algebra packages, in a single object, the stable multiplication rules of the class algebras of the symmetric groups.

If time permits, we will also discuss other settings in which Farahat-Higman-type phenomena appear. These include $GL(n, q)$, the Gelfand pair (B_n, S_{2n}), symplectic groups $Sp(2n, q)$, and wreath-product double-coset algebras. In particular, we shall mention Farahat-Higman-type constructions for $Sp(2n, q)$ and for the double-coset algebra associated with

$$(S_k \wr S_n,\, S_{kn}),$$

where the latter gives a noncommutative example. These examples show that the Farahat-Higman stability observation is not restricted to centers of symmetric group algebras; it also appears in Hecke-type and double-coset settings, sometimes beyond the commutative case.

A central point of the lecture is that the Farahat-Higman algebra is not obtained as a naive injective or projective limit of the centers $Z(\mathbb{Z}[S_n])$, leading to the following natural question: is there a more intrinsic limiting object whose multiplication explains the observed polynomial stability, whose answer will be given in the second lecture.

Lecture 2. Ivanov-Kerov partial permutations

The second lecture presents the Ivanov-Kerov construction, whose main idea is to enlarge permutations to partial permutations, thereby obtaining a "base" free interpretation of permutations. The partial permutations form a semigroups, and their semigroup algebras admit natural projection maps on the group algebras over permutations.

The symmetric group acts on partial permutations by conjugation, and the corresponding invariant subalgebras form a genuine projective system. We will show that the resulting projective-limit algebra is naturally isomorphic to the Farahat-Higman algebra.

Thus the Ivanov-Kerov construction gives a conceptual explanation of the Farahat-Higman algebra.

Like in the first lecture, we shall formulate the analogy with other families such as $GL(n, q)$, Méliot's algebra of partial isomorphisms may be viewed as an Ivanov-Kerov-type construction, while the work of Wan-Wang gives a Farahat–Higman-type stable center. The comparison between these constructions is a natural analogue of the Farahat-Higman/Ivanov-Kerov relationship for symmetric groups.

Lecture 3. From finite symmetric groups to S_∞: the Borodin-Olshanski route

The third lecture places the preceding stable algebraic constructions in the broader framework of asymptotic representation theory. In this lecture, we will summarize the first four chapters of Borodin-Olshanski. Namely, i- *finite symmetric groups*, ii- *symmetric functions*, iii- *coherent systems on the Young graph*, and iv- *the classification of extreme characters of* S_∞.

The starting point of this part is that central elements of $\mathbb{C}[S_n]$ act on irreducible representations by scalars. After normalization, these scalars become functions on Young diagrams. Okounkov and Olshanski identify the relevant stable algebra of such character functions with the algebra of shifted symmetric functions, a deformation of the symmetric functions, which appears as the FH and IK algebra of the family $(S_n)_{n\in\mathbb{N}}$. This gives the algebraic bridge from the Farahat-Higman/Ivanov-Kerov world to asymptotic character theory.

Having described the bridge, we will move on to the explanation of how coherent systems on the Young graph encode compatible sequences of probability measures on irreducible representations of S_n. The boundary of this graph describes limiting characters, and Thoma's theorem identifies the extreme characters of the infinite symmetric group S_∞ in terms of Thoma parameters.

The aim is not to give a complete proof of Thoma's theorem. Rather, the goal is to explain how the algebraic stability detected by Farahat-Higman and realized by Ivanov-Kerov fits into the representation-theoretic boundary picture developed by Borodin-Olshanski. In this sense, the third lecture we plan to supply the asymptotic representation-theoretic context for the universal algebraic structures constructed in the first two lectures.

The lecture concludes by returning to the broader motivation and open problems: for other families of groups and double-coset algebras, Farahat-Higman-type algebras may exist. Examples include $GL(n, q)$, $Sp(2n, q)$, the Gelfand pair (B_n, S_{2n}), and the noncommutative double-coset algebras associated with $(S_k \wr S_n, S_{kn})$. In several of these cases, the corresponding Ivanov-Kerov-type projective-limit constructions and Borodin-Olshanski/Okounkov-Olshanski-type asymptotic interpretations are either missing, or incomplete, at best, not yet understood.

Prerequisites

The course is aimed at graduate students and researchers with some background in representation theory and algebraic combinatorics.

1. **Essential background:** Finite groups, group algebras, conjugacy classes, characters, and basic representation theory of symmetric groups.
2. **Useful background:** Partitions, Young diagrams, class sums, and basic symmetric functions.
3. **Helpful but not required:** Projective limits, Hecke algebras, Gelfand pairs, and basic familiarity with asymptotic representation theory.

No prior knowledge of the full Okounkov–Olshanski or Borodin–Olshanski theory will be assumed. The necessary notions of shifted symmetric functions, normalized characters, coherent systems, Young graphs, and Thoma parameters will be introduced in the course.

Learning goals

After the mini-course, participants should be able to:

1. explain the polynomial stability phenomenon behind the Farahat-Higman construction;
2. describe why the Farahat-Higman algebra is not a naive limit of the centers $Z(\mathbb{Z}[S_n])$;

3. understand the Ivanov-Kerov algebra of partial permutations as an intrinsic projective-limit realization of the Farahat-Higman algebra;

4. explain how stable central elements and normalized characters lead to shifted symmetric functions;

5. describe the role of the Young graph, coherent systems, and Thoma parameters in the asymptotic representation theory of symmetric groups;

6. identify natural open problems about extending the Farahat-Higman/Ivanov-Kerov/Borodin–Olshanski picture beyond S_n.

Exercises and prerequisite sheets

Here is how this mini-course will work.

We will include short prerequisite sheets and some guided exercises. The sheets are really just there to jog your memory on things you have most probably seen before: conjugacy classes, class sums, Young subgroups, tableaux, tabloids, polytabloids, induced representations, and a bit of elementary symmetric functions. Nothing new, just a refresher.

The exercises are tied directly to those sheets, and they have two main purposes. First, I want you to get your hands dirty with the basic objects before we bring in the heavier abstract constructions. Second, some of the stability phenomena in the lectures only really click when you see them happening in concrete examples; the exercises are there to make that visible.

What kinds of exercises? Mostly manageable computations and guided arguments. For instance, we may compute products of conjugacy class sums in small symmetric groups, work out examples involving Young subgroups and induced characters, do some actual calculations with tabloids and polytabloids, or prove basic facts about symmetric functions in a guided way. Some exercises may also walk through standard results used in the lectures, such as the fundamental theorem of symmetric functions or the hook-length formula in small examples.

One thing I want to stress is that these materials are not here to add more prerequisites. The point is the opposite. They are meant to give you a concrete way into the subject. My hope is that, by working through these examples, the universal algebras in the lectures will feel less like black magic and more like ordinary mathematics.

Suggested reading

- H. K. Farahat and G. Higman, *The centres of symmetric group rings*, Proceedings of the Royal Society of London. Series A, 250 (1959).

- V. Ivanov and S. Kerov, *The algebra of conjugacy classes in symmetric groups, and partial permutations*, Zap. Nauchn. Sem. POMI 256 (1999).

- A. Okounkov and G. Olshanski, *Shifted Schur functions*, St. Petersburg Mathematical Journal 9 (1998).

- A. Borodin and G. Olshanski, *Representations of the infinite symmetric group*, Cambridge Studies in Advanced Mathematics, 2017.

- P.-L. Méliot, *Partial isomorphisms over finite fields.* J Algebr Comb 40, 83–136 (2014).

- W. Wang and J. Wan, *Stability of the centers of group algebras of* $GL(n,q)$, Adv. Math., 349 (2019), pp. 749-780.
- Ş. Özden, Stability of the centers of the symplectic group rings $Sp(2n,q)$, Journal of Algebra Volume 572, 2021, Pages 263-296.
- Ş. Özden, Stability of the Hecke algebra of wreath products, https://arxiv.org/abs/1910.10480